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Finite axisymmetric deformations of elastic tubes: An approximate method

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Abstract. Finite axisymmetric deformation of a hollow circular cylinder with a finite length, composed of a neo-Hookean material, is studied. The inner surface of the tube is subjected to both normal and tangential tractions, while the outer surface is free of tractions. The cylinder will undergo both radial and axial deformations. An asymptotic-expansion method is used to determine the stress and shape of the deformed tube. The deformed radial and axial coordinates, the stress tensor and the surface tractions are expanded into a power series of an appropriate thickness parameter. A hierarchy of equilibrium equations, boundary conditions and constitutive equation are derived following the usual procedure. The theories corresponding to the lowest two order members in this hierarchy are studied in detail. It is shown that the zeroth-order theory corresponds to the membrane theory. The shape of the deformed tube, up to the second-order in the thickness parameter, is determined in terms of the zeroth-order radial and axial deformations. The zeroth-order radial and axial deformations are governed by a coupled pair of nonlinear ordinary differential equations, both of which are of second order. For illustrative purposes the present approach is then applied to a simple representative problem: simultaneous extension and inflation of a cylindrical elastic tube. Finally, the solutions corresponding to the zeroth and first-order approximations of the present theory and the exact solutions obtained from finite elasticity theory are compared for the above-mentioned problem.

1. Introduction

Finite axisymmetric deformations of nonlinearly elastic circular tubes have been the subject of considerable interest in the last three decades or so. In general, it is very difficult to solve any problem encountered in practice, because of the nonlinearity of the equations involved. The so-called semi-inverse method is the technique most frequently used in the analysis of finite elasticity problems and its impact on the development of mechanics of non-metallic bodies has been tremendous. However, the number of problems for which exact solutions have been obtained by using the semi-inverse method are few (see [1,2,3], for example). In order to facilitate the calculations, most of the studies introduce some simplifying assumptions or approximations, i.e. they focus on membrane solutions and consider normal surface tractions only. However, in many practical problems the tangential and normal forces occur simultaneously on the same surface and the stresses vary through the thickness of the tube. A special type of deformation of such a problem, namely, finite axisymmetric deformations of a thin-walled elastic tube sliding on a rough rigid rod, has been studied by Kydoniefs and Spencer [4] in which an asymptotic expansion technique is used.

The general theory of axially symmetrical membranes is based on some simplifications which are usually assumed to be valid when the thickness of the tube is very small as compared to the inner radius of the tube. However, for many rubber-like and/or biological elastic tubes, these simplifying assumptions may not be valid. For instance, interest in finite axisymmetric deformation of cylindrical elastic tubes is growing, especially in the field of biomechanics, where the cylindrical tube structures, such as arteries, often undergo large elastic deformations [5]. Almost all the researchers working in the area of wave propagation in arteries have treated the arterial wall as thin-walled cylindrical shells or membranes. However, even for large arteries, the ratio of the thickness to the inner radius of the tube changes between 1/6 and 1/4. On the other hand, thin-shell or membrane theories are applicable when this ratio is much less than these values. Therefore, thin-shell or membrane theories in general cannot be applied to arterial mechanics. Furthermore, the problem of viscous flow inside an elastic tube, which provides a fairly good approximation for blood flow in arteries, must take into account both the tangential stress and axially varying internal pressure acting on the inner surface of the artery [6]. Due to the viscous drag, the pressure varies along any streamlines, even on the tube wall. That is, a viscous fluid flow through a cylindrical elastic tube will produce varying internal pressure and wall shear stress on the inner surface of the tube. The primary objective of this paper is to introduce a higher-order theory which takes into account both the variation of the stresses through the thickness and tangential as well as normal surface tractions.

In the present work, following the analysis and most of the notation of Kydoniefs and Spencer [4], a finite axisymmetric deformation of a hollow circular cylinder made of a neo-Hookean elastic material is considered. This material approximation is known to give qualitatively good agreement with experiment and is simple enough to get further insight in the nonlinear coupling which arises in the governing differential equations. The inner surface of the cylindrical tube is subjected to both normal and tangential tractions while the outer surface is free of tractions. In Section 2, the governing differential equations of such a hollow cylindrical shell made of neo-Hookean material are obtained in material coordinates. For its convenience in the analysis the field equations are expressed in terms of some non-dimensionalized quantities and a small thickness parameter ε of the tube. In Section 3, the asymptotic expansions of the field variables in terms of the thickness parameter ε lead to a hierarchy of the governing equations; and the equations corresponding to the first two order members of this hierarchy are studied in detail. It is shown that the zeroth-order theory corresponds to the membrane theory. In other words, the zeroth-order field variables are independent of the transverse (radial) coordinate. However, the first-order equations include the variations through the thickness. The zeroth- and first-order quantities are determined in terms of two functions of the undeformed axial coordinate. These two functions which represent the zeroth-order radial and axial deformations are governed by a coupled pair of nonlinear second-order ordinary differential equations. The equations given in [4] are obtained as a special case of the present derivation. In Section 4, for comparison of the present approximate results with the exact ones, one representative problem is considered: simultaneous extension and inflation of a cylindrical elastic tube. The three-dimensional finite elasticity solution to this problem is available in the literature and it is one of a few number of finite elasticity problems for which the exact solutions have been obtained. No specific problems, other than the above-mentioned example, are solved. However, the conclusions resulting from the comparison of approximate and exact solutions for this specific problem give a clear indication as to what can and what cannot be done with the present approach.

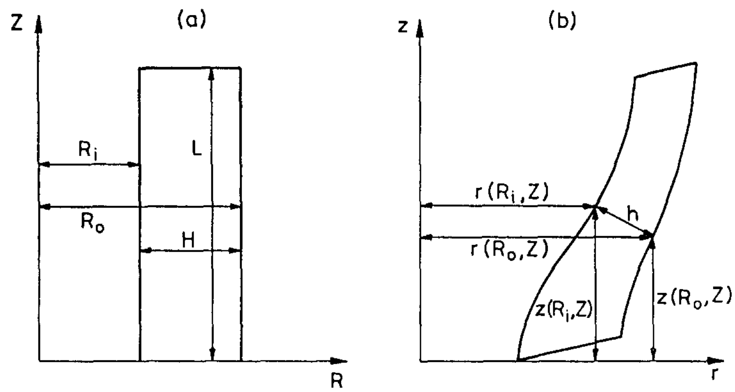

Fig. 1. Axial sections of the undeformed (a) and deformed (b) tube. Only the right half of the tube is shown.

2. Formulation of the problem

2.1. Deformation

Consider a hollow circular cylinder made of a neo-Hookean material, of inner radius R_i, outer radius R_o, thickness H and length L in its undeformed configuration[1], Fig. 1. The cylinder is subjected to both tangential and normal tractions on its inner surface while the outer surface is free of tractions. The surface tractions may vary axially, but not circumferentially so that an axisymmetric deformation is produced. Thus the deformation, which takes the point with cylindrical polar coordinates (R, Θ, Z) in the undeformed state to the point (r, θ, z) in the deformed state, has the form

$$r = r(R, Z), \quad \theta = \Theta, \quad z = z(R, Z). \tag{2.1}$$

Therefore, the inner surface of the deformed tube could be generated by rotating a continuous plane curve C about the z-axis. It is assumed that C has no double points and does not intersect the z-axis. Thus the curve C becomes a meridian curve on the inner surface of the deformed tube and lines of latitude are formed by the family of curves which are orthogonal to the meridians at every point. The inner and outer surfaces of the deformed tube have parametric equations $r = r(R_i, Z), z = z(R_i, Z)$ and $r = r(R_o, Z), z = z(R_o, Z)$, respectively. Then the variable "thickness" of the deformed tube is given by

$$h = \{[r(R_o, Z) - r(R_i, Z)]^2 + [z(R_o, Z) - z(R_i, Z)]^2\}^{\frac{1}{2}}, \tag{2.2}$$

and the tangential and normal unit vectors, $\mathbf{t}$ and $\mathbf{n}$, of the deformed inner surface $(R = R_i)$ are

$$\mathbf{t} = \frac{1}{\Phi}\left(\frac{\partial r}{\partial Z}\mathbf{e}_r + \frac{\partial z}{\partial Z}\mathbf{e}_z\right), \quad \mathbf{n} = -\frac{1}{\Phi}\left(\frac{\partial z}{\partial Z}\mathbf{e}_r - \frac{\partial r}{\partial Z}\mathbf{e}_z\right). \tag{2.3}$$

Here $\mathbf{e}_r$ and $\mathbf{e}_z$ are the radial and axial unit vectors in the cylindrical coordinates and $\Phi(R, Z)$ is defined by

$$\Phi(R,Z) = \left[\left(\frac{\partial r}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2\right]^{1/2}. \tag{2.4}$$

From the deformation field (2.1) the components of the deformation gradient tensor $\mathbf{F}$ and the components of the left Cauchy-Green (or Finger) deformation tensor $\mathbf{c}^{-1} = \mathbf{F}\mathbf{F}^T$ referred to cylindrical coordinates are defined as

$$\mathbf{F} = \begin{bmatrix} \frac{\partial r}{\partial R} & 0 & \frac{\partial r}{\partial Z} \\ 0 & \frac{r}{R} & 0 \\ \frac{\partial z}{\partial R} & 0 & \frac{\partial z}{\partial Z} \end{bmatrix}, \quad \mathbf{c}^{-1} = \begin{bmatrix} \left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2 & 0 & \frac{\partial r}{\partial R}\frac{\partial z}{\partial R} + \frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z} \\ 0 & \left(\frac{r}{R}\right)^2 & 0 \\ \frac{\partial r}{\partial R}\frac{\partial z}{\partial R} + \frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z} & 0 & \left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2 \end{bmatrix}. \tag{2.5}$$

The incompressibility of the material requires that

$$\det \mathbf{F} = \frac{r}{R}\left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z} - \frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right) = 1. \tag{2.6}$$

2.2. Constitutive Equations

In order to obtain the deformation field one needs the stress-strain relations for the cylindrical tube. From a practical point of view, there is often little to choose between various strain-energy functions (see, for example, [7]). The simpler the equation, the easier it will be to draw conclusions related to elastic behavior. Here it is assumed that the cylindrical tube is made of the neo-Hookean material which is widely used to describe the behavior of rubber-like materials. In order to simplify the presentation, attention is focused on this special case. This model is known to give qualitatively good agreement with experiment and is simple enough to allow further insight in the nonlinear coupling which arises in the governing equations. Henceforth, the Cauchy stress tensor σ is assumed to be

$$\sigma = -\tilde{p}\,\mathbf{I} + 2\alpha\,\mathbf{c}^{-1} \tag{2.7}$$

where $\mathbf{I}$ is the identity, $\tilde{p} = \tilde{p}(r, \theta, z)$ is the hydrostatic pressure to be determined from the field equations and the boundary conditions and α is a material constant which is to be determined from experimental measurements. For this particular problem the physical components of the symmetrical Cauchy stress tensor are given by

$$\begin{aligned} \sigma_{rr} &= -\tilde{p} + 2\alpha\left[\left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2\right], \quad \sigma_{rz} = 2\alpha\left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial R} + \frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z}\right) \\ \sigma_{\theta\theta} &= -\tilde{p} + 2\alpha\left(\frac{r}{R}\right)^2, \quad \sigma_{zz} = -\tilde{p} + 2\alpha\left[\left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2\right], \\ \sigma_{r\theta} &= \sigma_{\theta z} = 0 \end{aligned} \tag{2.8}$$

where the zero components are due to the assumed axial symmetry.

2.3. Equilibrium equations

In the absence of body forces, the equilibrium equations div $\boldsymbol{\sigma} = \mathbf{0}$ in the present case reduce to the following two equations

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{\partial \sigma_{rz}}{\partial z} = 0, \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\sigma_{rz} + \frac{\partial \sigma_{zz}}{\partial z} = 0. \tag{2.9}$$

The third equilibrium equation simply gives $\tilde{p} = \tilde{p}(r, z)$, i.e. $\tilde{p}$ is independent of θ. Furthermore, since the deformation is not known yet, it might be pertinent to express the equilibrium equations in the material coordinates (R, Z). If this is done, the following system of equilibrium equations is obtained.

$$\begin{aligned} &\frac{\partial}{\partial R}(r\sigma_{rr})\frac{\partial z}{\partial Z} - \frac{\partial}{\partial Z}(r\sigma_{rr})\frac{\partial z}{\partial R} + r\left(\frac{\partial r}{\partial R}\frac{\partial \sigma_{rz}}{\partial Z} - \frac{\partial r}{\partial Z}\frac{\partial \sigma_{rz}}{\partial R}\right) - \sigma_{\theta\theta}\left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z} - \frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right) = 0 \\ &\frac{\partial}{\partial R}(r\sigma_{rz})\frac{\partial z}{\partial Z} - \frac{\partial}{\partial Z}(r\sigma_{rz})\frac{\partial z}{\partial R} + r\left(\frac{\partial r}{\partial R}\frac{\partial \sigma_{zz}}{\partial Z} - \frac{\partial r}{\partial Z}\frac{\partial \sigma_{zz}}{\partial R}\right) = 0, \end{aligned} \tag{2.10}$$

where R and Z are taken as the independent variables. Note that the incompressibility condition (2.6) completes a system of the three equations for $r(R, Z)$, $z(R, Z)$ and $\tilde{p}(R, Z)$.

2.4. Boundary conditions

It is assumed that the cylinder is subjected to both tangential and normal tractions on its inner surface while the outer surface is free of tractions. Then the boundary conditions at the inner and outer surfaces of the deformed tube are given by

$$\left(\frac{\partial z}{\partial Z}\sigma_{rr} - \frac{\partial r}{\partial Z}\sigma_{rz}\right)/\Phi = -\tilde{P}_r, \quad \left(\frac{\partial z}{\partial Z}\sigma_{rz} - \frac{\partial r}{\partial Z}\sigma_{zz}\right)/\Phi = -\tilde{P}_z \ \text{ at } R = R_i \tag{2.11}$$

and

$$\frac{\partial z}{\partial Z}\sigma_{rr} - \frac{\partial r}{\partial Z}\sigma_{rz} = 0, \quad \frac{\partial z}{\partial Z}\sigma_{rz} - \frac{\partial r}{\partial Z}\sigma_{zz} = 0 \ \text{ at } R = R_o \tag{2.12}$$

respectively. Here $\tilde{P}_r(Z)$ and $\tilde{P}_z(Z)$ are the surface traction components on the inner surface $(R = R_i)$ of the deformed tube, in the radial and axial directions, respectively. In some cases, it might be more convenient to use the normal and tangential components of the surface traction, which are given by

$$\tilde{P}_n = \left(\frac{\partial z}{\partial Z}\tilde{P}_r - \frac{\partial r}{\partial Z}\tilde{P}_z\right)/\Phi, \quad \tilde{P}_t = \left(\frac{\partial r}{\partial Z}\tilde{P}_r + \frac{\partial z}{\partial Z}\tilde{P}_z\right)/\Phi \tag{2.13}$$

where $\tilde{P}_n(Z)$ and $\tilde{P}_t(Z)$ are the load per unit area in the normal and tangential directions of the inner surface of the deformed tube. For the time being, the boundary conditions at two ends of the cylinder will not be specified. This subject will be discussed later.

2.5. Non-dimensional variables

For the sake of simplicity in the subsequent asymptotic analysis, the basic equations will be expressed in non-dimensional form and for this purpose, the following non-dimensionalized quantities are introduced:

$$
\begin{aligned}
&\varepsilon = H/R_i, && A = L/R_i \\
&\eta = (R - R_i)/H \quad (0 \leqslant \eta \leqslant 1), && \zeta = Z/R_i \quad (0 \leqslant \zeta \leqslant A) \\
&u(\eta,\zeta) = r(R,Z)/R_i, && w(\eta,\zeta) = z(R,Z)/R_i \\
&\{\boldsymbol{\tau}, p, P_r, P_z, P_n, P_t\} = \tfrac{1}{2\alpha}\{\boldsymbol{\sigma}, \tilde{p}, \tilde{P}_r, \tilde{P}_z, \tilde{P}_n, \tilde{P}_t\}, &&
\end{aligned}
\tag{2.14}
$$

where ε and A are the thickness parameter and the aspect ratio of the cylinder, respectively. In this case the non-dimensional form of the function Φ is given by

$$
\Phi(\eta,\zeta) = \left[\left(\frac{\partial u}{\partial \zeta}\right)^2 + \left(\frac{\partial w}{\partial \zeta}\right)^2\right]^{\frac{1}{2}}, \tag{2.15}
$$

and the incompressibility condition takes the following form

$$
\frac{u}{\varepsilon(1+\varepsilon\eta)}\left(\frac{\partial u}{\partial \eta}\frac{\partial w}{\partial \zeta} - \frac{\partial u}{\partial \zeta}\frac{\partial w}{\partial \eta}\right) = 1. \tag{2.16}
$$

The non-zero stress components (2.8) in non-dimensional form become

$$
\begin{aligned}
\tau_{rr} &= -p + \frac{1}{\varepsilon^2}\left(\frac{\partial u}{\partial \eta}\right)^2 + \left(\frac{\partial u}{\partial \zeta}\right)^2, \quad && \tau_{rz} = \frac{1}{\varepsilon^2}\frac{\partial u}{\partial \eta}\frac{\partial w}{\partial \eta} + \frac{\partial u}{\partial \zeta}\frac{\partial w}{\partial \zeta} \\
\tau_{\theta\theta} &= -p + \left(\frac{u}{1+\varepsilon\eta}\right)^2, && \tau_{zz} = -p + \frac{1}{\varepsilon^2}\left(\frac{\partial w}{\partial \eta}\right)^2 + \left(\frac{\partial w}{\partial \zeta}\right)^2.
\end{aligned}
\tag{2.17}
$$

Hence the equilibrium equations take the following form

$$
\begin{aligned}
&\frac{\partial}{\partial \eta}(u\tau_{rr})\frac{\partial w}{\partial \zeta} - \frac{\partial}{\partial \zeta}(u\tau_{rr})\frac{\partial w}{\partial \eta} + u\left(\frac{\partial u}{\partial \eta}\frac{\partial \tau_{rz}}{\partial \zeta} - \frac{\partial u}{\partial \zeta}\frac{\partial \tau_{rz}}{\partial \eta}\right) - \tau_{\theta\theta}\left(\frac{\partial u}{\partial \eta}\frac{\partial w}{\partial \zeta} - \frac{\partial u}{\partial \zeta}\frac{\partial w}{\partial \eta}\right) = 0 \\
&\frac{\partial}{\partial \eta}(u\tau_{rz})\frac{\partial w}{\partial \zeta} - \frac{\partial}{\partial \zeta}(u\tau_{rz})\frac{\partial w}{\partial \eta} + u\left(\frac{\partial u}{\partial \eta}\frac{\partial \tau_{zz}}{\partial \zeta} - \frac{\partial u}{\partial \zeta}\frac{\partial \tau_{zz}}{\partial \eta}\right) = 0.
\end{aligned}
\tag{2.18}
$$

On the other hand, the boundary conditions (2.11)–(2.12), using the relations (2.14), can be expressed by

$$
\left(\frac{\partial w}{\partial \zeta}\tau_{rr} - \frac{\partial u}{\partial \zeta}\tau_{rz}\right)/\Phi = -P_r, \quad \left(\frac{\partial w}{\partial \zeta}\tau_{rz} - \frac{\partial u}{\partial \zeta}\tau_{zz}\right)/\Phi = -P_z \ \text{ at } \eta = 0 \tag{2.19}
$$

and

$$
\frac{\partial w}{\partial \zeta}\tau_{rr} - \frac{\partial u}{\partial \zeta}\tau_{rz} = 0, \quad \frac{\partial w}{\partial \zeta}\tau_{rz} - \frac{\partial u}{\partial \zeta}\tau_{zz} = 0 \ \text{ at } \eta = 1. \tag{2.20}
$$

In this case the relations (2.13) read

$$
P_n = \left(\frac{\partial w}{\partial \zeta}P_r - \frac{\partial u}{\partial \zeta}P_z\right)/\Phi, \quad P_t = \left(\frac{\partial u}{\partial \zeta}P_r + \frac{\partial w}{\partial \zeta}P_z\right)/\Phi. \tag{2.21}
$$

For reasons of simplicity in the subsequent analysis the following notation is introduced

$$u(0,\zeta) = g(\zeta), \quad w(0,\zeta) = f(\zeta). \tag{2.22}$$

As may be noticed, the functions $g(\zeta)$ and $f(\zeta)$ are nothing but the non-dimensional coordinates of the interior surface of the deformed tube. It will be further assumed that the inner surface of the tube does not intersect the axis of the symmetry, i.e. $g \neq 0$. Note that $g(\zeta) =$ constant in [4]. In other words, when the assumption $g(\zeta) =$ constant is made, the equations presented here agree with those in [4].

3. Asymptotic expansion

From the inspection of equations (2.15)–(2.22), it may be seen that it is almost impossible to find a closed-form solution to the field quantities. Therefore, following the analysis of Kydoniefs and Spencer [4], a regular perturbation solution based on the smallness of the thickness parameter ε is used in this work. As pointed out before, the thickness H will be assumed to be small as compared to the inner radius R_i, i.e. $\varepsilon = H/R_i \ll 1$. Furthermore, it will be assumed that all the field quantities can be expanded into a power series of the parameter ε, i.e.,

$$\begin{aligned} u(\eta,\zeta) &= g(\zeta) + \varepsilon u^{(1)}(\eta,\zeta) + \cdots \\ w(\eta,\zeta) &= f(\zeta) + \varepsilon w^{(1)}(\eta,\zeta) + \cdots \\ p(\eta,\zeta) &= p^{(0)}(\zeta) + \varepsilon p^{(1)}(\eta,\zeta) + \cdots \end{aligned} \tag{3.1}$$

where the coefficients of the powers of ε are assumed to be of order unity. Similarly, the stresses, surface tractions and function Φ may also be expanded as

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{(0)} + \varepsilon \boldsymbol{\tau}^{(1)} + \cdots, \quad P = P^{(0)} + \varepsilon P^{(1)} + \cdots, \quad \Phi = \phi + \varepsilon \Phi^{(1)} + \cdots \tag{3.2}$$

where P represents all the surface traction components, i.e. $P = (P_r, P_z, P_n, P_t)$ and the notation $\phi = \Phi^{(0)}$ has been used. Note that, as a consequence of the asymptotic expansion (i.e. η disappears from the equations as ε goes to zero), all the zeroth-order quantities are functions of ζ only. Introducing these expansions into the field equations and equating the coefficients of the powers of ε to zero, we can obtain successive systems of differential equations relating various terms in the expansion, but these will not all be listed here. In what follows, only the zeroth- and first-order solutions will be investigated.

3.1. Zeroth-order solution

Introducing (3.1) and (3.2) into (2.15)–(2.22), we obtain the following equations corresponding to the zeroth-order approximation. First, at the zeroth-order approximation, the function Φ given by (2.15) becomes

$$\phi(\zeta) = [(f')^2 + (g')^2]^{1/2} \tag{3.3}$$

where the prime denotes the derivative with respect to the non-dimensional axial coordinate ζ. The incompressibility condition (2.16), at the zeroth-order approximation, takes the following form

$$g\left(f'\frac{\partial u^{(1)}}{\partial \eta} - g'\frac{\partial w^{(1)}}{\partial \eta}\right) = 1 \tag{3.4}$$

which implies that $g \neq 0$. Similarly, from the constitutive relations given by (2.17), the following relations are valid

$$\begin{aligned} \tau_{rr}^{(0)} &= -p^{(0)} + \left(\frac{\partial u^{(1)}}{\partial \eta}\right)^2 + (g')^2, \quad \tau_{rz}^{(0)} = \frac{\partial u^{(1)}}{\partial \eta}\frac{\partial w^{(1)}}{\partial \eta} + f'g' \\ \tau_{\theta\theta}^{(0)} &= -p^{(0)} + g^2, \qquad\qquad \tau_{zz}^{(0)} = -p^{(0)} + \left(\frac{\partial w^{(1)}}{\partial \eta}\right)^2 + (f')^2. \end{aligned} \tag{3.5}$$

The equilibrium equations for the zeroth-order approximation read

$$f'\frac{\partial \tau_{rr}^{(0)}}{\partial \eta} - g'\frac{\partial \tau_{rz}^{(0)}}{\partial \eta} = 0, \quad f'\frac{\partial \tau_{rz}^{(0)}}{\partial \eta} - g'\frac{\partial \tau_{zz}^{(0)}}{\partial \eta} = 0. \tag{3.6}$$

Similarly, from (2.19) and (2.20), the boundary conditions become

$$(f'\tau_{rr}^{(0)} - g'\tau_{rz}^{(0)})/\phi = -P_r^{(0)}, \quad (f'\tau_{rz}^{(0)} - g'\tau_{zz}^{(0)})/\phi = -P_z^{(0)} \text{ at } \eta = 0 \tag{3.7}$$

and

$$f'\tau_{rr}^{(0)} - g'\tau_{rz}^{(0)} = 0, \quad f'\tau_{rz}^{(0)} - g'\tau_{zz}^{(0)} = 0 \text{ at } \eta = 1. \tag{3.8}$$

Noticing that (3.6) can be written as

$$\frac{\partial}{\partial \eta}(f'\tau_{rr}^{(0)} - g'\tau_{rz}^{(0)}) = 0, \quad \frac{\partial}{\partial \eta}(f'\tau_{rz}^{(0)} - g'\tau_{zz}^{(0)}) = 0 \tag{3.9}$$

and integrating these equations with respect to η, we have

$$f'\tau_{rr}^{(0)} - g'\tau_{rz}^{(0)} = c_1(\zeta), \quad f'\tau_{rz}^{(0)} - g'\tau_{zz}^{(0)} = c_2(\zeta). \tag{3.10}$$

Here $c_1(\zeta)$ and $c_2(\zeta)$ are two arbitrary functions to be determined from an application of the boundary conditions (3.8). Using these boundary conditions in (3.10), we have

$$c_1(\zeta) = c_2(\zeta) = 0 \tag{3.11}$$

Thus, the equilibrium equations (3.10), the boundary conditions (3.7) and the relations (2.21) lead to the following relations

$$f'\tau_{rr}^{(0)} - g'\tau_{rz}^{(0)} = 0, \quad f'\tau_{rz}^{(0)} - g'\tau_{zz}^{(0)} = 0 \tag{3.12}$$

and

$$P_r^{(0)} = P_z^{(0)} = P_n^{(0)} = P_t^{(0)} = 0. \tag{3.13}$$

Equation (3.13) shows that the surface tractions are of the order of $\varepsilon = H/R_i$. Moreover, from (2.22), the following relations are obtained

$$u^{(1)}(0,\zeta) = 0, \quad w^{(1)}(0,\zeta) = 0. \tag{3.14}$$

The incompressibility condition (3.4) and the equilibrium equations (3.12) give sufficient relations to determine the unknown functions $p^{(0)}, u^{(1)}$ and $w^{(1)}$. In order to solve these equations, the incompressibility condition (3.4) is first written as

$$\frac{\partial u^{(1)}}{\partial \eta} = \frac{1}{f'}\left(\frac{1}{g} + g'\frac{\partial w^{(1)}}{\partial \eta}\right). \tag{3.15}$$

Introducing (3.15) into (3.5) and then the results into (3.12), we obtain the following relations

$$p^{(0)} = \left(\frac{1}{g\phi}\right)^2, \quad \frac{\partial u^{(1)}}{\partial \eta} = \frac{f'}{g\phi^2}, \quad \frac{\partial w^{(1)}}{\partial \eta} = -\frac{g'}{g\phi^2}. \tag{3.16}$$

Since f, g and ϕ are independent of η, $(3.16)_{2,3}$ can be integrated with respect to η to obtain

$$u^{(1)} = \eta\frac{f'}{g\phi^2}, \quad w^{(1)} = -\eta\frac{g'}{g\phi^2}, \tag{3.17}$$

where the conditions (3.14) have been used. Thus the zeroth-order stress components take the following form

$$\begin{aligned}\tau^{(0)}_{rr} &= (g')^2\left(1 - \frac{1}{g^2\phi^4}\right), \quad \tau^{(0)}_{rz} = f'g'\left(1 - \frac{1}{g^2\phi^4}\right)\\ \tau^{(0)}_{\theta\theta} &= g^2\left(1 - \frac{1}{g^4\phi^2}\right), \quad \tau^{(0)}_{zz} = (f')^2\left(1 - \frac{1}{g^2\phi^4}\right).\end{aligned} \tag{3.18}$$

As is seen, all the zeroth-order quantities have been determined in terms of the functions $f(\zeta)$ and $g(\zeta)$ which will be determined by solving the related differential equations to be obtained from the first-order approximation.

3.2. First-order solution

In this subsection the first-order approximate solutions are obtained by using the expansion (3.1). Introducing (3.1) into (2.16) and using the zeroth-order results given by (3.17), we may express the first-order incompressibility condition as

$$f'\frac{\partial u^{(2)}}{\partial \eta} - g'\frac{\partial w^{(2)}}{\partial \eta} - \eta\Gamma_0(\zeta) = 0, \tag{3.19}$$

where

$$\Gamma_0(\zeta) = \frac{f'g'' - g'f''}{g^2\phi^4} - \frac{f'}{g^3\phi^2} + \frac{1}{g}. \tag{3.20}$$

Likewise, from (2.17) the first-order stress components take the following form

$$\begin{aligned}
\tau_{rr}^{(1)} &= -p^{(1)} + 2\left[\frac{f'}{g\phi^2}\frac{\partial u^{(2)}}{\partial\eta} + \eta g'\left(\frac{f'}{g\phi^2}\right)'\right]\\
\tau_{\theta\theta}^{(1)} &= -p^{(1)} + 2\eta\left(\frac{f'}{\phi^2} - g^2\right), \quad \tau_{zz}^{(1)} = -p^{(1)} - 2\left[\frac{g'}{g\phi^2}\frac{\partial w^{(2)}}{\partial\eta} + \eta f'\left(\frac{g'}{g\phi^2}\right)'\right]\\
\tau_{rz}^{(1)} &= \frac{1}{g\phi^2}\left(f'\frac{\partial w^{(2)}}{\partial\eta} - g'\frac{\partial u^{(2)}}{\partial\eta}\right) - \eta\left[g'\left(\frac{g'}{g\phi^2}\right)' - f'\left(\frac{f'}{g\phi^2}\right)'\right],
\end{aligned} \tag{3.21}$$

where the zeroth-order results given by (3.17) have been used. Furthermore, using (2.18) and (3.17), we see that the equilibrium equations corresponding to the first-order approximation of the expansion become

$$f'\frac{\partial\tau_{rr}^{(1)}}{\partial\eta} - g'\frac{\partial\tau_{rz}^{(1)}}{\partial\eta} + \Gamma_1(\zeta) = 0, \quad f'\frac{\partial\tau_{rz}^{(1)}}{\partial\eta} - g'\frac{\partial\tau_{zz}^{(1)}}{\partial\eta} + \Gamma_2(\zeta) = 0. \tag{3.22}$$

Here, the terms $\Gamma_1(\zeta)$ and $\Gamma_2(\zeta)$ which depend only on zeroth-order quantities are defined by

$$\begin{aligned}
\Gamma_1(\zeta) &= -g'\left[\Gamma_3(\zeta) + \frac{2}{g^2\phi^2}\left(\frac{1}{g\phi^2}\right)'\right] + \frac{g''}{g}\left(1 - \frac{1}{g^2\phi^4}\right) + \frac{1}{g^4\phi^2} - 1\\
\Gamma_2(\zeta) &= -f'\left[\Gamma_3(\zeta) + \frac{2}{g^2\phi^2}\left(\frac{1}{g\phi^2}\right)'\right] + \frac{f''}{g}\left(1 - \frac{1}{g^2\phi^4}\right)
\end{aligned} \tag{3.23}$$

with

$$\Gamma_3(\zeta) = \left(1 - \frac{1}{g^2\phi^4}\right)\left[\frac{f'f'' + g'g''}{g\phi^2} + \phi^2\left(\frac{1}{g\phi^2}\right)'\right]. \tag{3.24}$$

Similarly, using (3.17) and noticing that $\partial u^{(1)}/\partial\zeta = \partial w^{(1)}/\partial\zeta = 0$ at $\eta = 0$, we find that the first-order boundary conditions, from (2.19)–(2.20), take the following form

$$(f'\tau_{rr}^{(1)} - g'\tau_{rz}^{(1)})/\phi = -P_r^{(1)}, \quad (f'\tau_{rz}^{(1)} - g'\tau_{zz}^{(1)})/\phi = -P_z^{(1)} \quad \text{at } \eta = 0 \tag{3.25}$$

and

$$f'\tau_{rr}^{(1)} - g'\tau_{rz}^{(1)} = g'\Gamma_3(\zeta), \quad f'\tau_{rz}^{(1)} - g'\tau_{zz}^{(1)} = f'\Gamma_3(\zeta) \quad \text{at } \eta = 1. \tag{3.26}$$

On the other hand, the relations (2.21) in the first-order approximation reduce to

$$P_n^{(1)} = (f'P_r^{(1)} - g'P_z^{(1)})/\phi, \quad P_t^{(1)} = (g'P_r^{(1)} + f'P_z^{(1)})/\phi. \tag{3.27}$$

Finally, from (2.22), the following conditions for the first-order approximation are obtained

$$u^{(2)}(0,\zeta) = 0, \quad w^{(2)}(0,\zeta) = 0. \tag{3.28}$$

In order to solve the problem described above, as has been done for the zeroth-order approximation, first the equilibrium equations (3.22) are integrated with respect to η and the following equations are obtained

$$f'\tau_{rr}^{(1)} - g'\tau_{rz}^{(1)} + \eta\Gamma_1(\zeta) = C_1(\zeta), \quad f'\tau_{rz}^{(1)} - g'\tau_{zz}^{(1)} + \eta\Gamma_2(\zeta) = C_2(\zeta), \tag{3.29}$$

where $C_1(\zeta)$ and $C_2(\zeta)$ are two arbitrary functions to be determined. If the boundary conditions (3.26) are used, these functions are found in the following form

$$C_1(\zeta) = g'\Gamma_3(\zeta) + \Gamma_1(\zeta), \quad C_2(\zeta) = f'\Gamma_3(\zeta) + \Gamma_2(\zeta). \tag{3.30}$$

Using (3.29) and (3.30) in (3.25), we have

$$P_r^{(1)} = -(\Gamma_1 + g'\Gamma_3)/\phi, \quad P_z^{(1)} = -(\Gamma_2 + f'\Gamma_3)/\phi. \tag{3.31}$$

Similarly, $P_n^{(1)}$ and $P_t^{(1)}$ can be expressed as

$$P_n^{(1)} = -\frac{1}{\phi^2}(f'\Gamma_1 - g'\Gamma_2), \quad P_t^{(1)} = -\frac{1}{\phi^2}(f'\Gamma_2 + g'\Gamma_1) - \Gamma_3. \tag{3.32}$$

In order to determine the unknown functions $p^{(1)}$, $u^{(2)}$ and $w^{(2)}$, there exist three equations which are the incompressibility condition (3.19) and the equilibrium equations (3.29). Using (3.21) and (3.19) in (3.29) we can obtain two equations for the unknowns $p^{(1)}$ and $\partial w^{(2)}/\partial\eta$. After a lengthy manipulation the following results are found

$$p^{(1)} = \eta\psi_1 + \psi_2, \quad \frac{\partial u^{(2)}}{\partial\eta} = \eta\left(g'\psi_3 + \frac{\Gamma_0}{f'}\right) + g'\psi_4, \quad \frac{\partial w^{(2)}}{\partial\eta} = \eta f'\psi_3 + f'\psi_4 \tag{3.33}$$

where the functions $\psi_j(\zeta)(j = 1, 2, 3, 4)$ are defined by

$$\begin{aligned}
\psi_1(\zeta) &= \frac{1}{\phi^2}\left(\frac{2}{g}\Gamma_0 + f'\Gamma_1 - g'\Gamma_2\right), \quad \psi_2(\zeta) = \frac{-1}{\phi^2}(f'\Gamma_1 - g'\Gamma_2) \\
\psi_3(\zeta) &= -\frac{g}{\phi^2}\left[\frac{g'\Gamma_0}{gf'} + \phi^2 f'\left(\frac{f'}{g\phi^2}\right)' + \phi^2 g'\left(\frac{g'}{g\phi^2}\right)' + g'\Gamma_1 + f'\Gamma_2\right] \\
\psi_4(\zeta) &= \frac{g}{\phi^2}(g'\Gamma_1 + f'\Gamma_2 + \phi^2\Gamma_3).
\end{aligned} \tag{3.34}$$

Integrating $(3.33)_{2,3}$ with respect to η and using (3.28) one obtains $u^{(2)}$ and $w^{(2)}$ as

$$u^{(2)} = \frac{1}{2}\eta^2\left(g'\psi_3 + \frac{\Gamma_0}{f'}\right) + \eta g'\psi_4, \quad w^{(2)} = \frac{1}{2}\eta^2 f'\psi_3 + \eta f'\psi_4. \tag{3.35}$$

Finally, it is pointed out that using the expansion (3.1) and the above results in (2.2) we can calculate the deformed "thickness" up to the order of ε as follows

$$\frac{h}{H} = \frac{1}{g\phi} + \frac{\varepsilon}{2g\phi}\left(1 - \frac{f'}{g^2\phi^2} + \frac{f'g'' - g'f''}{g\phi^4}\right) + \cdots. \tag{3.36}$$

Note that $h/H = 1/g\phi$ at the zeroth-order approximation, which will be discussed later.

It is seen from these calculations that the non-dimensional deformed coordinates u and w are determined up to the order of ε^2 and the non-dimensional pressure function p and stress tensor τ are determined up to the order of ε, in terms of the functions f and g only. On the other hand, these functions can be determined from equations (3.31) or (3.32). Generally, considering that the normal and tangential components of the stresses on the boundary are given, it might be pertinent to use equations (3.32). Employing (3.23) and (3.24) in (3.32), the following system of two coupled second-order nonlinear ordinary differential equations is obtained

$$\begin{aligned} &f'g'' - g'f'' - f'\frac{\phi^2(1-g^4\phi^2)}{g(1-g^2\phi^4)} - \frac{g^3\phi^6}{1-g^2\phi^4}P_n^{(1)} = 0 \\ &f'f'' + g'g'' + g'\frac{\phi^2(3-g^4\phi^2)}{g(3+g^2\phi^4)} + \frac{g^3\phi^6}{3+g^2\phi^4}P_t^{(1)} = 0, \end{aligned} \tag{3.37}$$

where ϕ is given by (3.3). By introducing a new function $\beta(\zeta)$ and noting the relation (3.3), we may also write the above system in the following form, i.e. as a system of four coupled first-order nonlinear differential equations,

$$\begin{aligned} &f' = \phi\sin\beta, \quad g' = \phi\cos\beta \\ &\beta' + \frac{\phi(1-g^4\phi^2)}{g(1-g^2\phi^4)}\sin\beta + \frac{g^3\phi^4}{1-g^2\phi^4}P_n^{(1)}(\zeta) = 0 \\ &\phi' + \frac{\phi^2(3-g^4\phi^2)}{g(3+g^2\phi^4)}\cos\beta + \frac{g^3\phi^5}{3+g^2\phi^4}P_t^{(1)}(\zeta) = 0. \end{aligned} \tag{3.38}$$

Here $\beta(\zeta)$ is the angle between the tangent of the deformed interior generating line and the axis u in the (u, w) coordinate frame. These equations may also be written in terms of principal stretch ratios and principal stresses. For this purpose, in Appendix, the expressions of principal stretch ratios and principal stresses are first given within the context of the present theory and then, it is shown that equations (3.37) or (3.38) are in agreement with those derived in the membrane theory. For instance, equations (3.37) are identical to equations (39) and (40) of [8] when $P_t^{(1)}$ and $P_n^{(1)}$ are both zero and appropriate changes are made in the notation, i.e. the relations $\phi = \lambda_1$ and $g = \lambda_2$, derived in Appendix, are used.

Depending on the type of the boundary conditions, one of the systems (3.37) and (3.38) can be used to determine the deformation and stress field. The boundary conditions could be either in the form of specifying the functions f and g or their derivatives at one or both ends of the cylinder. If we can solve the above equations either analytically or numerically, using the relations obtained for the zeroth- and first-order quantities we can determine the non-dimensional deformed coordinates u and w up to the order of ε^2 and the non-dimensional pressure function p and stress tensor τ up to the order of ε. In other words, the solution of the system (3.37) or (3.38) makes it possible to reach easily the relationship between the undeformed and deformed profiles so as to include the variations (up to the quadratic powers of η) through the thickness.

A special case of the above equations are the equations derived by Kydoniefs and Spencer [4] for a thin-walled tube sliding on a rough rigid rod. This can be accomplished by substituting

$g(\zeta) = \lambda_\theta =$ constant. and consequently $g' = 0$ and $\phi = f'$ in the equations obtained here. Thus the system (3.37) and the boundary conditions (3.25) take the following form

$$P_n^{(1)}(\zeta) = \frac{\lambda_\theta^4 (f')^2 - 1}{\lambda_\theta^4 (f')^3}, \quad P_t^{(1)}(\zeta) = -\frac{3 + \lambda_\theta^2 (f')^4}{\lambda_\theta^3 (f')^5} f'' \tag{3.39}$$

and

$$-\tau_{rr}^{(1)} = P_n^{(1)}, \quad -\tau_{rz}^{(1)} = P_t^{(1)} \text{ at } \eta = 0 \tag{3.40}$$

respectively. Thus, as in [4], imposing the condition $\tau_{rz}^{(1)} = -k\tau_{rr}^{(1)}$ (or $P_t^{(1)} = -kP_n^{(1)}$) at $\eta = 0$, where k is a given constant, we reach the following second-order differential equation

$$f'' = k \frac{[\lambda_\theta^4 (f')^2 - 1](f')^2}{\lambda_\theta [3 + \lambda_\theta^2 (f')^4]} \tag{3.41}$$

in accordance with equation (5.11) of Kydoniefs and Spencer [4]. Equation (3.41) is solved in [4] under the boundary conditions

$$f(0) = 0, \quad f'(0) = 1/\sqrt{\lambda_\theta}. \tag{3.42}$$

Whereas the first boundary condition corresponds to the case where there is no axial deformation at $\zeta = 0$, the second one corresponds to the case where the surface $\zeta = 0$ is free of external forces, i.e. the case where $\tau_{zz}^{(0)} = 0$ at $\zeta = 0$ and the requirement that $f(\zeta)$ is an increasing function of ζ for small ζ is satisfied. Thus, the equations of Kydoniefs and Spencer [4] are recovered by the present study.

As mentioned in [9] and [10], two types of problem can be defined. In the first problem one specifies the normal and tangential surface tractions $P_n^{(1)}(\zeta)$ and $P_t^{(1)}(\zeta)$ and tries to find the functions f and g under appropriately defined end conditions. In the second problem the functions f and g are specified, and the surface tractions $P_n^{(1)}(\zeta)$ and $P_t^{(1)}(\zeta)$ are determined so as to satisfy the above differential equations. As has been pointed out before, f and g are the coordinates of the inner surface of the deformed tube. Thus, prescribing f and g means that we specify the inner surface of the tube after deformation. This case arises if, for example, the deformed tube encloses a rigid axisymmetric body of a given shape. The first problem may be described as a direct problem, whereas the second one is a semi-inverse problem. The problems of the first kind are more difficult than those of the second, since we need to solve the coupled highly nonlinear equations given by (3.37) or (3.38). The task of obtaining analytical solutions to the system (3.37) or (3.38), even for the case where $P_t^{(1)}(\zeta) \equiv 0$, is formidable. Various recent papers (see [11], for example) describe different numerical approaches to solve an equivalent pair of the present equations for the case of $P_t^{(1)}(\zeta) \equiv 0$. To solve the above system, (3.37) or (3.38), of the coupled highly nonlinear differential equations numerically and a detailed discussion of the solutions obtained is not within the scope of the present paper. Moreover, the edge effects resulting from the finite length of the tube are not discussed here, but it is hoped that we may do so in a later paper.

In the remainder of this paper we would like to examine the accuracy of the approximate equations derived here through a simple problem, comparing the results of the present theory with exact solutions. However, the number of finite elasticity problems for which exact solutions have been obtained is small. The mathematics involved is often extremely complex and apparently many simple problems within the context of finite elasticity theory have not yet been solved in closed form. Therefore, the method presented here is applied to a typical problem with a known closed-form solution, namely, the combined extension and inflation of a cylindrical tube.

4. An example and comparison of results of finite elasticity and present theory

4.1. Present Theory

This section examines the accuracy of the approximate equations derived in the previous section for a specific problem. The present approach will be validated through comparison of the results of the present theory with exact three-dimensional solutions from the theory of finite elasticity. The limited number of known exact solutions in finite elasticity, however, severely restricts comparison with the present theory. The example presented here merely serves the purpose of comparing the zeroth-order, first-order and exact solutions. It is assumed that the curve C which generates the inner surface of the deformed tube by rotating it about the z-axis, is a vertical line in the (r, z) coordinate frame. In this case, the function g becomes equal to a constant, i.e. $g(\zeta) = \lambda_\theta = \text{constant}$. Consequently $g' = 0$ and $\lambda_1 = f'$, i.e. f' represents the stretch ratio in the axial direction. Thus, from (3.37) we again obtain the relations given by (3.39). If f' is assumed to be equal to a constant, i.e. $f' = \lambda_z = \text{constant.}$, then (3.39) reduces to

$$P_n^{(1)} = \frac{\lambda_\theta^4\lambda_z^2 - 1}{\lambda_\theta^4\lambda_z^3}, \quad P_t^{(1)} = 0. \tag{4.1}$$

In this case, from (3.16)–(3.18), the zeroth-order approximation for this specific problem becomes

$$\begin{aligned} g &= \lambda_\theta, \quad f = \lambda_z\zeta \\ p^{(0)} &= \frac{1}{\lambda_\theta^2\lambda_z^2}, \quad u^{(1)} = \frac{\eta}{\lambda_\theta\lambda_z}, \quad w^{(1)} = 0 \\ \tau_{rr}^{(0)} &= \tau_{rz}^{(0)} = 0, \quad \tau_{\theta\theta}^{(0)} = \lambda_\theta^2 - \frac{1}{\lambda_\theta^2\lambda_z^2}, \quad \tau_{zz}^{(0)} = \lambda_z^2 - \frac{1}{\lambda_\theta^2\lambda_z^2}. \end{aligned} \tag{4.2}$$

Similarly, from (3.21), (3.33) and (3.35), the first-order approximation takes the following form

$$\begin{aligned} p^{(1)} &= \frac{\eta}{\lambda_z}\left(\frac{2}{\lambda_\theta^2\lambda_z} - \frac{1}{\lambda_\theta^4\lambda_z^2} - 1\right) + \frac{1}{\lambda_z}\left(1 - \frac{1}{\lambda_\theta^4\lambda_z^2}\right) \\ u^{(2)} &= \frac{\eta^2}{2\lambda_\theta\lambda_z}\left(1 - \frac{1}{\lambda_\theta^2\lambda_z}\right), \quad w^{(2)} = 0 \end{aligned}$$

$$\tau_{rr}^{(1)} = \frac{\eta - 1}{\lambda_z}\left(1 - \frac{1}{\lambda_\theta^4 \lambda_z^2}\right), \quad \tau_{rz}^{(1)} = 0, \quad \tau_{zz}^{(1)} = -p^{(1)}$$

$$\tau_{\theta\theta}^{(1)} = \frac{\eta}{\lambda_z}\left[3 + \frac{1}{\lambda_\theta^4 \lambda_z^2} - \frac{2}{\lambda_\theta^2 \lambda_z}(1 + \lambda_\theta^4 \lambda_z^2)\right] - \frac{1}{\lambda_z}\left(1 - \frac{1}{\lambda_\theta^4 \lambda_z^2}\right). \tag{4.3}$$

Moreover, the deformed "thickness" becomes

$$\frac{h}{H} = \frac{1}{\lambda_\theta \lambda_z} + \frac{\varepsilon}{2\lambda_\theta \lambda_z}\left(1 - \frac{1}{\lambda_\theta^2 \lambda_z}\right) + \cdots \tag{4.4}$$

As will be shown later, the stresses given in equations (4.2) and (4.3) correspond to stresses of a circular tube subjected to a simultaneous extension and inflation. That is, after deformation the tube preserves its circular cylindrical form. Since the load is a uniform pressure, the tangential external load $P_t^{(1)}$ is not present in this problem, and the normal load $P_n^{(1)}$ is a constant. This completes the stress analysis of the problem within the context of the present theory.

4.2. Finite elasticity theory

The three-dimensional solution to the above-mentioned problem can be found in a number of publications on finite elasticity (e.g. [1, 2, 3]) and will not be rederived here. Here we summarize the exact solutions corresponding to the problem of an initially cylindrical tube made of neo-Hookean material, subjected to an axial tensile force and internal pressure. Thus the dimensionless forms of the normal load P_n, the hydrostatic pressure p and the stresses τ for this problem turn out to be

$$P_n = \frac{1}{\lambda_z^2}\left[\frac{1}{2}\left(\frac{1}{\Lambda_{2o}^2} - \frac{1}{\Lambda_{2i}^2}\right) + \lambda_z \ln\left(\frac{\Lambda_{2i}}{\Lambda_{2o}}\right)\right],$$

$$p = \frac{1}{\lambda_z^2}\left[\frac{1}{2}\left(\frac{1}{\Lambda_2^2} + \frac{1}{\Lambda_{2o}^2}\right) - \lambda_z \ln\left(\frac{\Lambda_{2o}}{\Lambda_2}\right)\right],$$

$$\tau_{rr} = \frac{1}{\lambda_z^2}\left[\frac{1}{2}\left(\frac{1}{\Lambda_2^2} - \frac{1}{\Lambda_{2o}^2}\right) + \lambda_z \ln\left(\frac{\Lambda_{2o}}{\Lambda_2}\right)\right],$$

$$\tau_{\theta\theta} = \Lambda_2^2 - \frac{1}{\lambda_z^2}\left[\frac{1}{2}\left(\frac{1}{\Lambda_2^2} + \frac{1}{\Lambda_{2o}^2}\right) - \lambda_z \ln\left(\frac{\Lambda_{2o}}{\Lambda_2}\right)\right],$$

$$\tau_{zz} = \lambda_z^2 - \frac{1}{\lambda_z^2}\left[\frac{1}{2}\left(\frac{1}{\Lambda_2^2} + \frac{1}{\Lambda_{2o}^2}\right) - \lambda_z \ln\left(\frac{\Lambda_{2o}}{\Lambda_2}\right)\right], \tag{4.5}$$

where the subscripts (i) and (o) stand for the values of a quantity evaluated on the inner and outer surfaces of the tube, respectively. The stresses can also be written in the concise form

$$\tau_{rr} = -p + \frac{1}{\lambda_z^2 \Lambda_2^2}, \quad \tau_{\theta\theta} = -p + \Lambda_2^2, \quad \tau_{zz} = -p + \lambda_z^2 \tag{4.6}$$

Moreover, the ratio of the thickness of the deformed tube to the thickness of the undeformed one is given by

$$\frac{h}{H} = \frac{1}{\varepsilon}\left[-\lambda_\theta + \sqrt{\lambda_\theta^2 + \frac{\varepsilon}{\lambda_z}(2+\varepsilon)}\right]. \tag{4.7}$$

It can be easily shown that the results given by (4.2) and (4.3), which correspond to the zeroth- and first-order approximations of the present theory, are identical to the first two terms of the series expansion of the above exact solution with respect to ε. In other words, the forms of the preceeding solution become identical to those of the present theory if we introduce

$$\Lambda_{2i} = \lambda_\theta, \quad \Lambda_2 = \lambda_\theta + \frac{\eta}{\lambda_z\lambda_\theta}(1 - \lambda_z\lambda_\theta^2)\varepsilon + \cdots, \quad \Lambda_{2o} = \lambda_\theta - \lambda_\theta\left(1 - \frac{1}{\lambda_z\lambda_\theta^2}\right)\varepsilon + \cdots \tag{4.8}$$

into (4.5) or (4.6) and neglect the higher-order terms than ε^2.

4.3. Results

The results presented in this paragraph test the limitations of the present theory compared to the finite elasticity theory and the advantages of the present theory compared to the membrane theory. For a comparison of the exact results, (4.5)–(4.7), with the approximate ones, (4.1)–(4.4), the influence of the thickness parameter ε and the Cauchy–stress distributions across the thickness of the tube were numerically examined for the above-mentioned problem. In all the calculations presented here we assume that $\lambda_\theta = 1.5$ and $\lambda_z = 1.5$, which are the stretch ratios defined at the inner surface of the deformed tube and correspond to those in the circumferential and axial directions, respectively. Furthermore, in all the following figures the long and short dashes correspond to the zeroth-order (i.e. membrane theory) and first-order approximations of the present theory, respectively and the full line to the finite elasticity theory.

In Fig. 2, h/H which represents the ratio of the deformed thickness to the undeformed thickness of the tube, is plotted against the thickness parameter ε for the cases of the membrane theory, the present theory and the finite elasticity theory. It is seen that the membrane theory is not valid if the thickness of the undeformed tube is comparable with the inner radius of the undeformed tube. It is also seen that the larger ε, the greater the differences between the results corresponding to the present theory and finite elasticity theory. In other words, accuracy improves with decreasing ε. Fig. 3 shows the variation of Λ_2 versus η for various values of ε. It is seen that Λ_2, which is the principal stretch ratio in the circumferential direction, is a monotonically decreasing function of the dimensionless radial coordinate η for all values of ε. The curves in Figs. 2 and 3 reveal that the membrane theory is inadequate even for very thin cylinders, whereas the accuracy within the context of the present theory remains acceptable as long as cylinders, at least as thick as $\varepsilon = H/R_i > 1/10$, are not considered.

Figs. 4, 5 and 6 show the Cauchy–stress distributions, τ_{rr}, $\tau_{\theta\theta}$ and τ_{zz}, respectively, across the thickness for three different values of ε. The stresses computed according to the current extended theory are extremely accurate, showing noticeable deviations from the finite elasticity solution only when we consider relatively thick cylinders ($\varepsilon > 1/10$). As shown by these figures, disregarding thickness effects we find substantial errors in all parts of the tube wall.

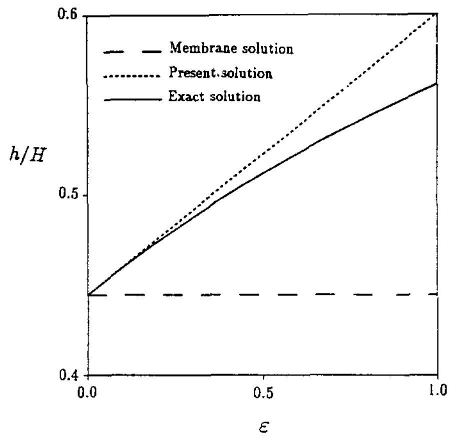

Fig. 2. Variation of h/H with ε for $\lambda_\theta = \lambda_z = 1.5$.

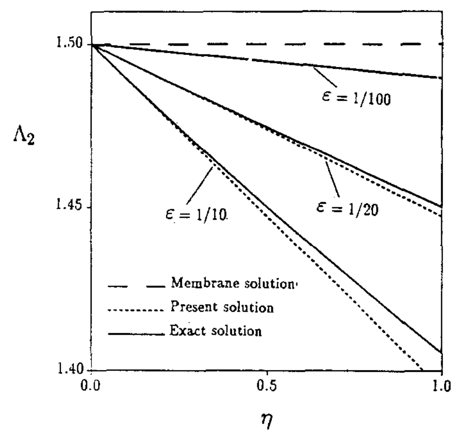

Fig. 3. Variation of Λ_2 through thickness for $\lambda_\theta = \lambda_z = 1.5$.

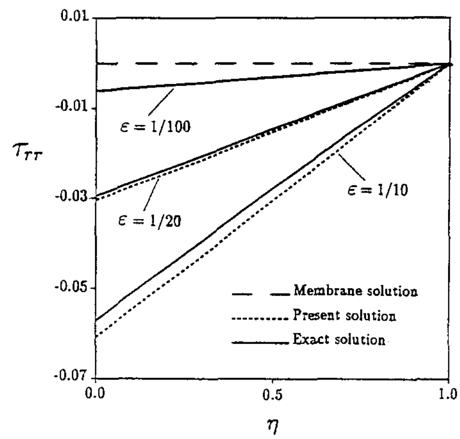

Fig. 4. Variation of τ_{rr} through thickness for $\lambda_\theta = \lambda_z = 1.5$.

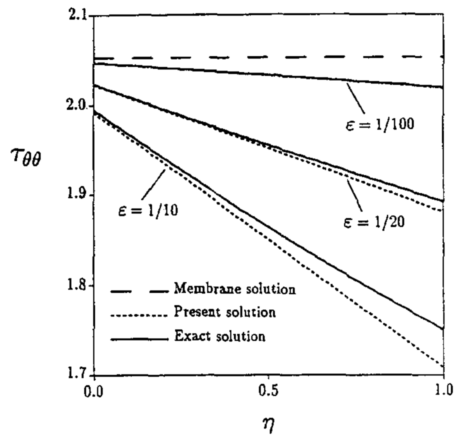

Fig. 5. Variation of $\tau_{\theta\theta}$ through thickness for $\lambda_\theta = \lambda_z = 1.5$.

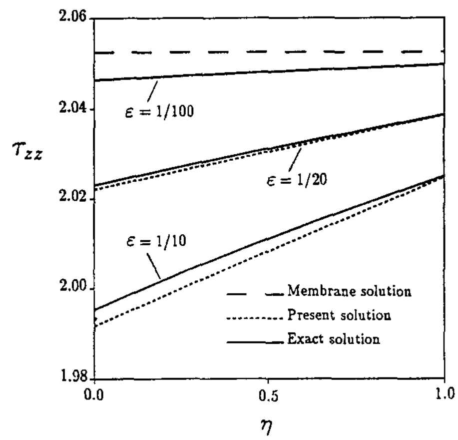

Fig. 6. Variation of τ_{zz} through thickness for $\lambda_\theta = \lambda_z = 1.5$.

These discrepancies, however, decrease as the tube becomes thinner. The results of this section suggest two main conclusions: *(i)* The extended theory presented here can give more accurate stresses for relatively thick tubes compared with those obtained from the membrane theory. *(ii)* For the tubes that are thin ($\varepsilon \leqslant 1/10$) the contributions from the finite elasticity theory to the present theory can usually be neglected.

5. Conclusions

In this paper, a method that takes into account both the variations of stresses through the thickness and tangential as well as normal surface tractions has been described for finite axisymmetric deformations of an initially cylindrical elastic tube. Using the asymptotic expansion technique, we have reformulated the problem of large axisymmetric deformations of nonlinear elastic membranes and have shown that the first-order approximate solutions may rapidly be obtained if the membrane solutions are available. The ratio of the constant thickness H of the undeformed tube to its inner radius is assumed to be small enough to justify the use of a perturbation approach in this problem. Through a simple representative problem it was shown that the present theory gives more accurate stresses compared with those obtained from the membrane theory and that for thin tubes the contributions from the finite elasticity theory to the present theory can be neglected.

It should be noted from (3.18) and (4.2) that the zeroth-order stress components, $\tau_{rr}^{(0)}$ and $\tau_{rz}^{(0)}$, are not zero in general, whereas they are identically equal to zero for the specific problem considered here. This shows that, in general, the non-dimensional stress components τ_{rr} and τ_{rz} are not of the order of ε as it is proposed by Kydoniefs and Spencer in [4]. This observation may provide the basis for a general theory of axisymmetric membranes with tangential tractions.

In the present work, no attempt was made to introduce the boundary conditions defined at one or both ends of the tube and to match the solutions to these boundary conditions, but it is hoped that we may do so in a later paper.

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Appendix

For comparison purposes, it may be more convenient to express the equations derived here in terms of principal stretch ratios and principal stresses. As is known, the squares of the principal stretch ratios (Λ_1, Λ_2 and Λ_3) are the eigenvalues of the relevant right Cauchy–Green (or Green) deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Thus, using $(2.5)_1$ to calculate the components of the right Cauchy–Green deformation tensor referred to cylindrical coordinates, we see that the principal stretch ratios Λ_1 and Λ_3 should be calculated as the solutions of the following equation

$$\left[\left(\frac{\partial r}{\partial R}\right)^2+\left(\frac{\partial z}{\partial R}\right)^2-\Lambda^2\right]\left[\left(\frac{\partial r}{\partial Z}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2-\Lambda^2\right]-\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}+\frac{\partial z}{\partial R}\frac{\partial z}{\partial Z}\right)^2=0, \tag{A1}$$

with $\Lambda_2 = r/R$, which is the principal stretch ratio in the circumferential direction. In other words, we have

$$\Lambda_1^2+\Lambda_3^2=\left(\frac{\partial r}{\partial R}\right)^2+\left(\frac{\partial z}{\partial R}\right)^2+\left(\frac{\partial r}{\partial Z}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2, \quad \Lambda_1^2\Lambda_3^2=\left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z}-\frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right)^2. \tag{A2}$$

Note that the material incompressibility condition $\Lambda_1\Lambda_2\Lambda_3 = 1$ gives the same as that in (2.6). Introducing the non-dimensional quantities defined by (2.14), we may write the above results in the form

$$\left\{\frac{1}{\varepsilon^2}\left[\left(\frac{\partial u}{\partial \eta}\right)^2+\left(\frac{\partial w}{\partial \eta}\right)^2\right]-\Lambda^2\right\}\left[\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial w}{\partial \zeta}\right)^2-\Lambda^2\right]$$
$$-\frac{1}{\varepsilon^2}\left(\frac{\partial u}{\partial \eta}\frac{\partial u}{\partial \zeta}+\frac{\partial w}{\partial \eta}\frac{\partial w}{\partial \zeta}\right)^2=0 \tag{A3}$$

and $\Lambda_2 = u/(1+\varepsilon\eta)$. The principal stretches ratios $\Lambda_k(k=1,2,3)$ may also be expanded as

$$\Lambda_k(\eta,\zeta)=\lambda_k(\zeta)+\varepsilon\Lambda_k^{(1)}(\eta,\zeta)+\cdots,\quad (k=1,2,3), \tag{A4}$$

where the notation $\lambda_k = \Lambda_k^{(0)}(k=1,2,3)$ has been used and λ_1, λ_2 and λ_3 correspond to the principal stretch ratios defined at the inner surface of the deformed tube. Introducing (3.1) into (A3) and using (3.17), the principal stretch ratios corresponding to the zeroth- and first-order approximations can be calculated. Thus, at the zeroth-order approximation, the principal stretch ratios take the following form

$$\lambda_1(\zeta)=\phi=[(f')^2+(g')^2]^{1/2},\quad \lambda_2(\zeta)=g,\quad \lambda_3(\zeta)=\frac{1}{\lambda_1\lambda_2}, \tag{A5}$$

which are identical to those obtained from membrane theory [2, 8, 9, 10, 11] and correspond to the principal stretch ratios defined along the meridian, the lines of latitude and the normal to the deformed surface, respectively. Similarly, at the first-order approximation we obtain the following results

$$\Lambda_1^{(1)} = \eta\frac{g'f''-f'g''}{g\phi^3},\quad \Lambda_2^{(1)}=\eta g\left(\frac{f'}{g^2\phi^2}-1\right)$$
$$\Lambda_3^{(1)} = \frac{\eta}{g\phi}\left(\frac{f'g''-g'f''}{g\phi^4}-\frac{f'}{g^2\phi^2}+1\right)=-\frac{\lambda_1\Lambda_2^{(1)}+\lambda_2\Lambda_1^{(1)}}{\lambda_1^2\lambda_2^2}, \tag{A6}$$

which can also be written in the concise form

$$\Lambda_1^{(1)}=\eta\frac{\kappa_1}{\lambda_2},\quad \Lambda_2^{(1)}=\eta\lambda_2\left(\frac{\kappa_2}{\lambda_1\lambda_2}-1\right),\quad \Lambda_3^{(1)}=\frac{\eta}{\lambda_1\lambda_2}\left(1-\frac{\kappa_1}{\lambda_1\lambda_2}-\frac{\kappa_2}{\lambda_1\lambda_2}\right), \tag{A7}$$

where κ_1 and κ_2 are the principal curvatures of the inner surface and are defined by

$$\kappa_1=\frac{g'f''-f'g''}{\lambda_1^3},\quad \kappa_2=\frac{f'}{\lambda_1\lambda_2}. \tag{A8}$$

As is known, the principal stresses (τ_1, τ_2 and τ_3 in dimensionless form) are the eigenvalues of the Cauchy stress tensor. Thus, noting that $\tau_{r\theta}=\tau_{z\theta}=0$, we see that the dimensionless principal stresses τ_1 and τ_3 should be calculated as the solutions of the following equation

$$(\tau_{rr}-\tau)(\tau_{zz}-\tau)-\tau_{rz}^2=0, \tag{A9}$$

with $\tau_2=\tau_{\theta\theta}$, which is the principal stress in the circumferential direction. Introducing the expansion (3.2) into (A9) and expanding the principal stresses as follows

$$\tau_k(\eta,\zeta)=\tau_k^{(0)}(\zeta)+\varepsilon\tau_k^{(1)}(\eta,\zeta)+\cdots\quad (k=1,2,3), \tag{A10}$$

the principal stresses corresponding to the zeroth- and first-order approximations can be calculated. Thus, at the zeroth-order approximation, the principal stresses are found as

$$\tau_1^{(0)} = \tau_{rr}^{(0)} + \tau_{zz}^{(0)}, \quad \tau_2^{(0)} = \tau_{\theta\theta}^{(0)}, \quad \tau_3^{(0)} = 0 \tag{A11}$$

or, using (3.18)

$$\tau_1^{(0)} = \lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2}, \quad \tau_2^{(0)} = \lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2}, \quad \tau_3^{(0)} = 0, \tag{A12}$$

which can also be written in the concise form

$$\tau_1^{(0)} = \lambda_1^2 - \lambda_3^2, \quad \tau_2^{(0)} = \lambda_2^2 - \lambda_3^2, \quad \tau_3^{(0)} = 0. \tag{A13}$$

Similarly, at the first-order approximation we obtain

$$\begin{aligned} \tau_1^{(1)} &= \frac{1}{\phi^2}[(f')^2\tau_{zz}^{(1)} + (g')^2\tau_{rr}^{(1)} + 2f'g'\tau_{rz}^{(1)}], \quad \tau_2^{(1)} = \tau_{\theta\theta}^{(1)}, \\ \tau_3^{(1)} &= \frac{1}{\phi^2}[(g')^2\tau_{zz}^{(1)} + (f')^2\tau_{rr}^{(1)} - 2f'g'\tau_{rz}^{(1)}] \end{aligned} \tag{A14}$$

or, using (3.21)

$$\begin{aligned} \tau_1^{(1)} &= -p^{(1)} - 2\eta\left[f'\left(\frac{g'}{g\phi^2}\right)' - g'\left(\frac{f'}{g\phi^2}\right)'\right], \quad \tau_2^{(1)} = -p^{(1)} + 2\eta\left(\frac{f'}{\phi^2} - g^2\right), \\ \tau_3^{(1)} &= -p^{(1)} + \frac{2}{g\phi^2}\left(f'\frac{\partial u^{(2)}}{\partial\eta} - g'\frac{\partial w^{(2)}}{\partial\eta}\right), \end{aligned} \tag{A15}$$

which can also be written in the concise form

$$\tau_1^{(1)} = -p^{(1)} + 2\lambda_1\Lambda_1^{(1)}, \quad \tau_2^{(1)} = -p^{(1)} + 2\lambda_2\Lambda_2^{(1)}, \quad \tau_3^{(1)} = -p^{(1)} + 2\lambda_3\Lambda_3^{(1)}. \tag{A16}$$

In this case, note that equation $(3.37)_1$ can be written in the following form $P_n^{(1)} = -\tau_3^{(1)}$ at $\eta = 0$, where $(A14)_3$, (3.25) and (3.27) are used.

We now want to show that the equations presented here for the zeroth-order approximation are identical to those derived by using membrane theory. If we consider the ratio of the thickness of the undeformed tube to the thickness of the deformed one for the zeroth-order approximation, we find from (3.36) and (A5) that $\lambda_3 = h/H$ which is a well-known relation in membrane theory [2, 9, 10, 11]. In order to express the principal stresses at the zeroth-order approximation as the principal stress resultants (forces per unit length), we should multiply them by h. In other words, the non-dimensional principal stress resultants are defined as

$$T_k = \frac{h}{H}\tau_k^{(0)} \left(\text{or} \quad T_k = \lambda_3\tau_k^{(0)} = \frac{1}{\lambda_1\lambda_2}\tau_k^{(0)}\right) \quad (k = 1, 2, 3) \tag{A17}$$

at the zeroth-order approximation. Thus, if (A12) is introduced into (A17), T_1 and T_2, which are the dimensionless principal stress resultants in the directions of the meridian curves and the curves of latitude, respectively, become

$$T_1 = \frac{\lambda_1^4\lambda_2^2 - 1}{\lambda_1^3\lambda_2^3}, \quad T_2 = \frac{\lambda_1^2\lambda_2^4 - 1}{\lambda_1^3\lambda_2^3} \tag{A18}$$

at the zeroth-order approximation. These results are identical to those derived for initially cylindrical membranes by using membrane theory [2, 8, 9, 10, 11]. Furthermore, using (A5) and (A18), after some calculations the system (3.37) can be expressed in terms of T_1 and T_2 as

$$(\lambda_2 T_1)' - \lambda_2' T_2 + \lambda_1 \lambda_2 P_t^{(1)} = 0$$
$$\kappa_1 T_1 + \kappa_2 T_2 - P_n^{(1)} = 0. \tag{A19}$$

In general, the equations given by (A19) are equivalent to those derived by using membrane theory. For instance, in the absence of a tangential traction component, i.e. $P_t^{(1)} = 0$, they are identical to equations (4.11.10) of [2].

Notes

[1] The notation of Kydoniefs and Spencer [4] will be followed whenever possible.

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